IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Some group-theoretical aspects of the $SO_0(1,4)$-invariant theory

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1988 J. Phys. A: Math. Gen. 21 599

(http://iopscience.iop.org/0305-4470/21/3/016)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 31/05/2010 at 15:35

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 21 (1988) 599-615. Printed in the UK

Some group-theoretical aspects of the $SO_0(1, 4)$-invariant theory

F M Lev
North-Eastern Interdisciplinary Research Institute, Magadan, USSR

Received 17 November 1986, in final form 9 September 1987

Abstract. Some properties of unitary representations of the de Sitter group $SO_0(1,4)$ are considered. In the case of spin zero and positivity of the mass operator, these representations involve a decomposition which ensures the fulfilment of the cluster separability condition and allows for the conventional interpretation of the de Sitter analogues of momentum and boost operators. The decomposition of the tensor product of representations belonging to the principal series contains the representations of the principal series with all masses. Under natural assumptions concerning the interaction operators, there are no bound states in the theory, which is due to the universal de Sitter antigravity.

1. Introduction

The theories in which the kinematic invariance group is not the Poincaré group but one of the de Sitter groups $SO(1,4)$ or $SO(2,3)$ have been considered by many physicists. From the group-theoretical and aesthetic points of view, the de Sitter invariance looks much more attractive than the Poincaré invariance. However, recently the de Sitter invariance has been studied less intensively, since in the currently popular superstring theories the flatness of the spacetime is supposed from the beginning.

The superstring theory has caused many physicists to contemplate that the existence of the covariant Lagrangian is unnecessary and that the correct physical properties should be imposed only on the representation of the invariance group or algebra in the corresponding Hilbert space. Previously, such a point of view was the basis for the construction of relativistic theories of systems with a fixed number of degrees of freedom and of more general non-local theories in the works by Sokolov (1977, 1978), Coester and Polyzou (1982), Mutze (1984) and others. Though these works used different techniques, the so-called Sokolov method of packing operators (Sokolov 1977, 1978) was the basis for them.

In our preceding paper (Lev 1984) we have proposed a Poincaré-invariant formulation of the packing operators method such that it is not essential whether the interaction is local or non-local and that the number of particles may vary. The analogous problem can be discussed in the case when the invariance group is not the Poincaré group but some other group (or supergroup) G. Briefly, the problem is as follows. One has to construct a unitary representation of the group G, $g \to U(g)$, satisfying the following condition: if $\{\alpha_1, \ldots, \alpha_n\}$ is any set of subsystems comprising the considered system, then upon elimination of interactions between these subsystems the above representation is transformed into the tensor product of representations $g \to U_{\alpha_i}(g)$ describing the corresponding subsystems. Such a property of the representation is called the cluster separability property, first considered by Haag and Kastler (1964).

 © 1988 IOP Publishing Ltd

The main purpose of the present paper is the extension of the author's results (Lev 1984) to the case when the invariance group is the de Sitter group $SO_0(1,4)$. It will be clear from the following discussion what reasons have prompted the author to try first the case $G = SO_0(1,4)$ and not $G = SO_0(2,3)$. The motivation for combining the de Sitter invariance with non-locality had been proposed to the author by Mirmovich (Lev and Mirmovich 1984).

As we shall see below, proceeding from the algebraic consideration which uses only the commutation relations for the representation generators of the group $G = SO_0(1,4)$ Lie algebra, one can note that the $SO_0(1,4)$-invariant theory has essentially different properties from the Poincaré-invariant theory.

Let us describe now the main notions and notations used in the present work. We denote by $(\ldots,\ldots)$ and $\|\ldots\|$ the scalar product and the norm in the considered Hilbert space. If $\mathscr{O}$ is an operator in H, then $\mathscr{D}(\mathscr{O})$ and $\bar{\mathscr{O}}$ denote its domain and closure respectively. Only the continuous unitary representations of the group G and its subgroups are considered. If not stated otherwise, summation over the repeated indices is assumed. In our notation the indices $a, b, c, d, e, f = 0, 1, \ldots, 4$, the indices $i, j = 1, \ldots, 4$, the indices $k, l, m = 1, \ldots, 3$, and the indices $\rho, \sigma = 1, 2$.

Having a representation $g \to U(g)$, one can construct the representation of the Lie algebra in the space of infinitely differentiable vectors H_∞. Let the matrices L^{ab} be as follows:

$$(L^{ab})^c_d = \delta^a_d \eta^{bc} - \delta^b_d \eta^{ac} \tag{1.1}$$

where the indices c and d enumerate the lines and columns, and η is the diagonal metric tensor with the components $\eta^{00} = -\eta^{11} = \ldots = -\eta^{44} = 1$. The representation generators M^{ab} are defined by the formula

$$\mathrm{i}M^{ab}x = \lim_{t\to 0} \frac{1}{t}\{U[\exp(tL^{ab})] - 1\}x \tag{1.2}$$

if $x \in H_\infty$. These operators are essentially self-adjoint in H_∞ and satisfy the commutation relations

$$[M^{ab}, M^{cd}] = -\mathrm{i}(\eta^{ac}M^{bd} + \eta^{bd}M^{ac} - \eta^{ad}M^{bc} - \eta^{bc}M^{ad}). \tag{1.3}$$

As usual, η_{ab} means the covariant tensor with the same components as η^{ab}, and both tensors are used to lower and raise the indices.

Let $\mathscr{A}$ be the algebra of polynomials of M^{ab}. The involution * on $\mathscr{A}$ is defined as an antilinear operator with the property

$$(M^{ab}M^{cd}\ldots M^{ef})^* = M^{ef}\ldots M^{cd}M^{ab}.$$

Theorem 1.1. If $x, y \in H_\infty$, $E(M) \in \mathscr{A}$ then $(x, E(M)y) = (E(M)^*x, y)$.

The theorem easily follows from the hermiticity of M^{ab} in H_∞ (see, e.g., proposition 2 from ch 11, § 1, of the book by Barut and Raczka (1977)).

Let $K_2 = M_{ij}M_{ij}$ be the Casimir operator of second order for the group $SO_0(4)$. Since $SO_0(4)$ is the maximum compact group in G and G is semisimple, we have the following theorem.

Theorem 1.2. If $E(M) \in \mathscr{A}$, $E(M)^* = E(M)$ and $[E(M), K_2] = 0$, then $E(M)$ is an essentially self-adjoint operator.

This theorem is a particular case of corollary 2 of theorem 3 from ch 11, § 2 of Barut and Raczka (1977). The last condition of theorem 1.2 is needed to ensure the self-adjointness of $E(M)$ in H_∞ (the hermiticity being ensured by theorem 1.1).

2. Generators of single-particle representations of the group $SO_0(1,4)$

In this section, some properties of unitary irreducible representations (UIR) of the group $SO_0(1,4)$ are briefly discussed. The complete classification of such representations was given by Dixmier (1961) and their various properties have been studied in many works (see, e.g., Strom (1970), Mensky (1976), Moylan (1983, 1985) and the references quoted therein).

Let A be the Abelian subgroup generated by L^{04} and T be the Abelian subgroup generated by the elements $L^{0k}+L^{4k}$. We consider the representation of the group $H = SO_0(3)AT$ defined by the formula

$$\Delta_{\mu s}(r\tau_{A}\boldsymbol{a}_T) = \exp(i\mu\tau)\Delta_s(r) \tag{2.1}$$

where $r \to \Delta_s(r)$ is the UIR of the group $SO_0(3)$ with the spin s, $\tau_A = \exp(\tau L^{04})$ and $\boldsymbol{a}_T$ is arbitrary (the reduction of $\Delta_{\mu s}$ on T is the identity representation). Let $g \to U(g)$ be the representation of the group G induced from the representation (2.1). The explicit form of operators M^{ab} depends on the choice of representatives for the elements of coset space G/H. The two choices are widely used. In the first of them the elements v_L and $v_L I$ are chosen as representatives, where v_L are the Lorentz group elements and I is the matrix formally coincident with the matrix η. In this case the representation under consideration is realised in the space of vector functions $\{\varphi_1(v), \varphi_2(v)\}$ where v is the element of the Lorentz velocity hyperboloid corresponding to v_L while the functions $\varphi_1(v)$ and $\varphi_2(v)$ have the range in the space of the UIR $r \to \Delta_s(r)$. The explicit calculation shows that the action of generators on the function $\varphi_1(v)$ is as follows:

$$\begin{gathered} \boldsymbol{M} = \boldsymbol{l}(\boldsymbol{v}) + \boldsymbol{s} \qquad \boldsymbol{N} = -i v_0 \frac{\partial}{\partial \boldsymbol{v}} + \frac{\boldsymbol{s}\times\boldsymbol{v}}{v_0+1} \\ \boldsymbol{F} = \mu\boldsymbol{v} + i\left[\frac{\partial}{\partial\boldsymbol{v}} + \boldsymbol{v}\left(\boldsymbol{v}\frac{\partial}{\partial\boldsymbol{v}}\right) + \frac{3}{2}\boldsymbol{v}\right] + \frac{\boldsymbol{s}\times\boldsymbol{v}}{v_0+1} \\ M_{04} = \mu v_0 + i v_0\left(\boldsymbol{v}\frac{\partial}{\partial\boldsymbol{v}} + \frac{3}{2}\right) \end{gathered} \tag{2.2}$$

where $\boldsymbol{M} = \{M^{23}, M^{31}, M^{12}\}$, $\boldsymbol{N} = \{M^{01}, M^{02}, M^{03}\}$, $\boldsymbol{F} = \{M^1{}_4, M^2{}_4, M^3{}_4\}$, $\boldsymbol{s}$ is the spin operator, $\boldsymbol{l}(\boldsymbol{v}) = -i\boldsymbol{v}\times\partial/\partial\boldsymbol{v}$ is the operator of orbital angular momentum and $v_0 = (1+\boldsymbol{v}^2)^{1/2}$.

If $\mu > 0$ and the functions $\{\varphi_1(v), \varphi_2(v)\}$ satisfy the condition

$$\int \{\|\varphi_1(v)\|^2 + \|\varphi_2(v)\|^2\}\, dv < \infty \tag{2.3}$$

where dv is the Lorentz invariant volume element on the hyperboloid, then we have a principal series of UIR describing the elementary particles. In the case $G = SO_0(2,3)$, however, the elementary particles are described by the UIR of discrete series (see, e.g., Evans (1967) and Fronsdal (1965)).

The above choice of representatives is utilised to obtain the possible closest analogy between the representations of $SO_0(1,4)$ and the Poincaré group. In particular, the operators $\boldsymbol{M}$ and $\boldsymbol{N}$ have the same form for both groups, while the contraction of $SO_0(1,4)$ into the Poincaré group is accomplished as follows. Let us introduce the parameter R and define $m=\mu/R$, $\boldsymbol{P}=\boldsymbol{F}/R$, $E=M_{04}/R$. Then, it follows from (2.2) that at $R\to\infty$, $\boldsymbol{P}=m\boldsymbol{v}$, $E=mv_0$, i.e. $\boldsymbol{P}$ and E are the conventional momentum and energy operators, respectively, and m is the conventional mass.

Since $G=SO_0(4)AT$, then one can choose as the representative x_G of the class x_GH the element x_G which is the representative of the class $x_G SO_0(3)$ in $SO_0(4)$. Rigorously speaking, we must deal not with $SO_0(1,4)$ but with its singly-connected covering group $\overline{SO_0(1,4)}$. The explicit description of $\overline{SO_0(1,4)}$ can be found elsewhere (see, e.g., Strom (1970)). The maximum compact subgroup of $\overline{SO_0(1,4)}$ is isomorphic to $K=SU(2)\times SU(2)$, where $\times$ denotes a direct product. The elements of the group K are denoted as (u,v) $(u,v\in SU(2))$. Let K_0 be the subgroup of K consisting of the elements (u,u) and L be the subgroup consisting of the elements $(u,1)$. At the covering homomorphism of the group K onto $SO_0(4)$ the subgroup K_0 is transformed into $SO_0(3)$. Therefore, the role of x_G can be played by the elements of the group L, and $K=L\otimes K_0$ where $\otimes$ means a semidirect product.

Let the elements of the group SU(2) be represented by the points of a three-dimensional sphere S^3 in four-dimensional space by means of the relation $u(n)=n_4+\mathrm{i}\boldsymbol{\sigma n}$, where $\{\boldsymbol{\sigma}\}$ are the Pauli matrices and $n_4=\pm(1-\boldsymbol{n}^2)^{1/2}$ for the upper and lower hemispheres respectively. Then, the UIR under consideration is realised in the space of vector functions on S^3 with the range in the space of UIR of the group SU(2) with the spin s. The direct calculation gives

$$\boldsymbol{L}=\frac{1}{2}\left(\mathrm{i}n_4\frac{\partial}{\partial\boldsymbol{n}}+\boldsymbol{l}(\boldsymbol{n})\right)\qquad \boldsymbol{M}=\boldsymbol{l}(\boldsymbol{n})+\boldsymbol{s}$$

$$\boldsymbol{N}=\mathrm{i}\left[\frac{\partial}{\partial\boldsymbol{n}}-\boldsymbol{n}\left(\boldsymbol{n}\frac{\partial}{\partial\boldsymbol{n}}\right)\right]-(\mu+\tfrac{3}{2}\mathrm{i})\boldsymbol{n}+\boldsymbol{n}\times\boldsymbol{s}-n_4\boldsymbol{s} \tag{2.4}$$

$$M_{04}=\mathrm{i}n_4\boldsymbol{n}\frac{\partial}{\partial\boldsymbol{n}}+(\mu+\tfrac{3}{2}\mathrm{i})n_4-\boldsymbol{sn}$$

where $\boldsymbol{L}$ are the representation generators of the group L. They are related to the operator $\boldsymbol{F}$ via the relationship $\boldsymbol{L}=\frac{1}{2}(\boldsymbol{F}+\boldsymbol{M})$.

The principal series corresponds to the case when $\mu>0$ and

$$\int \|\varphi(n)\|^2\,\mathrm{d}n<\infty \tag{2.5}$$

where $\mathrm{d}n$ is the SO(4)-invariant volume element on S^3

The above choice of representatives will be used in the main part of the work, while the first choice will be used only in § 6. A detailed description of various choices is given in the book by Klimyk and Kachurik (1986).

Let $n_K=(u(n),1)$ and kn be the 4-vector obtained from the 4-vector $n\in S^3$ by SO(4) rotation corresponding to $k\in K$. Then $(k|n)\equiv kn_K(kn)_K^{-1}$ is the element of the group K_0 and the generators $\boldsymbol{L}$ and $\boldsymbol{M}$ in (2.4) correspond to the following representation of the group K

$$U(k)\varphi(n)=\Delta_s[(k^{-1}|n)^{-1}]\varphi(k^{-1}n). \tag{2.6}$$

This is the representation of the group K induced from the representation $k_0 \rightarrow \Delta_s(k_0)$ of the group K_0. If $l \in L$, then, as follows from (2.6), $U(l)\varphi(n) = \varphi(l^{-1}n)$. Hence the representation operators of the group L do not act over the spin variables and produce only the left shift of the variable n. Therefore, S^3 can be considered as the de Sitter analogue of the coordinate space and $\boldsymbol{L}$ as the de Sitter analogue of momentum. The analogy with momentum follows also from the fact that the contraction of $\overline{SO_0(1,4)}$ into the Poincaré group transforms $K = L \otimes K_0$ into $\overline{E(3)} = T^3 \otimes K_0$, where T^3 is the group of translations of three-dimensional Euclidean space. However, the components of 'momentum' $\boldsymbol{L}$ do not commute with each other and satisfy the commutation relations for the representation generators of the group SU(2).

The sphere S^3 is considered in the literature (see, e.g., Mensky (1976), Fabec (1977) and Moylan (1983, 1985)) as a velocity space, since the class $v_L H$ coincides with the class $n_K H$ with $\boldsymbol{n} = -\boldsymbol{v}/v_0$, $n_4 = 1/v_0$, the generators of group T being in this case the de Sitter analogues of the momentum operator. Our further considerations are not formally dependent on whether we deal with the sphere S^3 as a velocity or coordinate space. However, by its meaning it corresponds to the latter case.

3. Representation operators of the group K in the multiparticle case

In the author's previous work (Lev 1984) it is shown that in the Poincaré-invariant theory one can easily ensure the cluster separability property in the so-called instant form when the representation generators of the group $\overline{E(3)}$ are free of interaction. Therefore, if we extend the results of this work to the case $G = SO_0(1, 4)$, it is natural to consider a version of the theory such that the representation generators of the group K are free of interaction. In this case the representation of the group K describing the system with a fixed number of elementary particles is the tensor product of the corresponding single-particle representations. Hence we can define the representation of the group K for any quantum system, since its representation space is a direct sum of spaces with a given number of particles.

We consider only the systems of particles corresponding to the principal series of UIR. Hence, we assume that N elementary particles are available which are indexed by $1, 2, \ldots, N$. Their states are described by the wavefunctions $\varphi(n_1, \ldots, n_N)$ with the range in the tensor product $\Delta_{s_1} \otimes \ldots \otimes \Delta_{s_N}$, being such that

$$\int \| \varphi(n_1, \ldots, n_N) \|^2 \, dn_1 \ldots dn_N < \infty. \tag{3.1}$$

We introduce the variable describing the motion of the system as a whole and the relative motion of particles. We choose the following as a unit 4-vector describing the system as a whole:

$$n = \frac{n_1 + \ldots + n_N}{|n_1 + \ldots + n_N|} \tag{3.2}$$

where the modulus of the 4-vector has its usual sense. We choose as a variable describing the 'internal' motion of the particles the unit 4-vector $\tilde{n}_i$ such that $(n_i)_K = n_K(\tilde{n}_i)_K$. Then from the definition of elements n_K and from (3.2) one can show that

$$\tilde{\boldsymbol{n}}_1 + \ldots + \tilde{\boldsymbol{n}}_N = 0 \qquad \tilde{n}_1^4 + \ldots + \tilde{n}_N^4 = |n_1 + \ldots + n_N| \geq 0. \tag{3.3}$$

Furthermore, a calculation similar to that in relativistic kinematics gives

$$\mathrm{d}n_1 \ldots \mathrm{d}n_N = \mathrm{d}n\, \mathrm{d}\delta(\mathrm{int}) \tag{3.4}$$

$$\mathrm{d}\delta(\mathrm{int}) = 2(\tilde{n}_1^4 + \ldots + \tilde{n}_N^4)\delta^{(3)}(\tilde{\boldsymbol{n}}_1 + \ldots + \tilde{\boldsymbol{n}}_N)\theta(\tilde{n}_1^4 + \ldots + \tilde{n}_N^4)\, \mathrm{d}\tilde{n}_1 \ldots \mathrm{d}\tilde{n}_N. \tag{3.5}$$

Using the formulae of this section and the formula (2.6), one can show that the considered N-particle representation of the group K can be given by

$$U(k)\psi(n, \mathrm{int}) = \Delta((k^{-1}|n)^{-1})\psi(k^{-1}n, \mathrm{int}) \tag{3.6}$$

where the function ψ depends on the vector n and the 'internal' variables to which we assign the vectors $\tilde{n}_1, \ldots, \tilde{n}_N$ and the spin variables of all particles. In addition, the function $\psi(n, \mathrm{int})$ satisfies the condition

$$\int \| \psi(n, \mathrm{int}) \|^2 \, \mathrm{d}n\, \mathrm{d}\delta(\mathrm{int}) < \infty \tag{3.7}$$

and the representation $k_0 \rightarrow \Delta(k_0)$ of the group K_0 entering the formula (3.6) is defined as a representation acting in the space of functions from the 'internal' variables as follows:

$$\Delta(k_0)\chi(\tilde{n}_1, \ldots, \tilde{n}_N) = \left(\prod_{i=1}^{N} \Delta_{s_i}(k_0)\right)\chi(k_0^{-1}\tilde{n}_1, \ldots, k_0^{-1}\tilde{n}_N). \tag{3.8}$$

The function $\chi(\tilde{n}_1, \ldots, \tilde{n}_N)$ entering this formula has the range in $\Delta_{s_1} \otimes \ldots \otimes \Delta_{s_N}$ and satisfies the condition

$$\int \| \chi(\tilde{n}_1, \ldots, \tilde{n}_N) \|^2 \, \mathrm{d}\delta(\mathrm{int}) < \infty. \tag{3.9}$$

From formula (3.8) it readily follows that the generators of the representation $k_0 \rightarrow \Delta(k_0)$ have a standard form of the total internal angular momentum operator

$$\boldsymbol{S} = \boldsymbol{l}_1(\tilde{\boldsymbol{n}}_1) + \ldots + \boldsymbol{l}_{N-1}(\tilde{\boldsymbol{n}}_{N-1}) + \boldsymbol{s}_1 + \ldots + \boldsymbol{s}_N. \tag{3.10}$$

The comparison of formulae (2.6) and (3.6) shows now that the generators of the N-particle representation of the group K have the form of single-particle representation generators (2.4) if n_i, $\boldsymbol{s}_i$ are replaced with n and $\boldsymbol{S}$, respectively. Note, however, that this is not the case for the generators of non-compact transformations even if the interaction between particles is absent.

Our choice of the de Sitter analogue of the momentum operator implies that $\boldsymbol{L}^2$ is the analogue of $\boldsymbol{P}^2$. Its eigenvalues, however, are now discrete and equal to $J(J+1)$, $J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, and the projector onto the corresponding states is

$$\prod{}^J = \frac{1}{2J+1} \int \mathrm{tr}\{\mathscr{D}^J(l)\} U(l)\, \mathrm{d}l \tag{3.11}$$

where $\mathscr{D}^J(l)$ is the matrix of the UIR of the group SU(2) with spin J, and $\mathrm{d}l$ is the invariant volume element on the group $\mathrm{L} \sim \mathrm{SU}(2)$ coinciding with the volume element of S^3.

In the conventional case, the momentum of the system at the fixed $\boldsymbol{P}^2$ is defined by two additional parameters fixing the spatial angle. In our case we can assume that at the fixed J the 'momentum' is defined by two parameters $\alpha, \beta = -J, -J+1, \ldots, J$ such that the wavefunction in the state $|J\alpha\beta\rangle$ is equal to

$$\psi^J_{\alpha\beta}(n, \mathrm{int}) = \mathscr{D}^J_{\alpha\beta}(n)\chi^J_{\alpha\beta}(\mathrm{int}) \tag{3.12}$$

where the sphere S^3 is identified with the space of the group SU(2), and $\chi^J_{\alpha\beta}$ is some function of the 'internal' variables. Thus, at a given J there are $(2J+1)^2$ different 'momenta', and all corresponding spaces $H^J_{\alpha\beta}$ have the same dimension equal to that of the space H^0.

4. Cluster separability in a system with spin zero

In order to extend the results of the author's work (Lev 1984) to the case $G = SO_0(1, 4)$, one has to determine the conditions at which, having the representation $g \to U(g)$ in the Hilbert space H, one can construct the unitary operators $\mathcal{U}^J_{\alpha\beta}$ from H^0 to $H^J_{\alpha\beta}$. Considering this problem, we shall use essentially the technique of $SU(2) \times SU(2)$ shift operators developed by Hughes (1983). Hence, for the reader's convenience, we change the notation in accordance with this paper. Instead of $\boldsymbol{L}$ we now write $\boldsymbol{P}$. This notation occasionally confirms that this operator is chosen as the de Sitter analogue of momentum. Certainly, one can choose as such an analogue the representation generators of the second multiplier in $SU(2) \times SU(2)$, and we denote them by $\boldsymbol{Q}$. Hence

$$\boldsymbol{P} = \tfrac{1}{2}(\boldsymbol{M} + \boldsymbol{F}) \qquad \boldsymbol{Q} = \tfrac{1}{2}(\boldsymbol{M} - \boldsymbol{F}). \tag{4.1}$$

Let us introduce $p_0 = P^3$, $p_\pm = P^1 \pm \mathrm{i} P^2$ and, analogously, introduce q_0, $q_\pm$. We introduce also the $SU(2) \times SU(2)$ spinor operator $R_{\rho\sigma}$ corresponding to the SO(4) vector operator M^{0i}

$$\begin{aligned} R_{11} &= 2^{-1/2}(\mathrm{i}M^{01} - M^{02}) \qquad & R_{12} &= 2^{-1/2}(-M^{04} - \mathrm{i}M^{03}) \\ R_{21} &= 2^{-1/2}(M^{04} - \mathrm{i}M^{03}) \qquad & R_{22} &= 2^{-1/2}(-\mathrm{i}M^{01} - M^{02}). \end{aligned} \tag{4.2}$$

We mean that the spinorial indices are raised and lowered with the help of antisymmetric spinor $\varepsilon_{\rho\rho'}$, where $\varepsilon_{12} = 1$. Then, from theorem 1.1 it readily follows that for any $x, y \in H_\infty$

$$(R_{\rho\sigma}x, y) = (x, R^{\rho\sigma}y). \tag{4.3}$$

Proceeding from (1.3) and (4.1), one can easily confirm that the commutation relations in the variables $(p_\pm, p_0, q_\pm, q_0, R_{\rho\sigma})$ are the same as those for the group SO(5) in Hughes' (1983) paper; the difference lies only in the hermiticity property (4.3). Therefore, the Casimir operators

$$I_2 = \tfrac{1}{4} M_{ab} M^{ab} \qquad I_4 = \tfrac{1}{64} V^a V_a \tag{4.4}$$

where $V^a = e^{abcde} M_{bc} M_{de}$ and e^{abcde} is the absolutely antisymmetric tensor with $e^{01234} = 1$, have the same form as in Hughes (1983). Since there is an inaccuracy in formula (3.4) of Hughes' work, we introduce an accurate expression for I_4:

$$\begin{aligned} I_4 = {} & 2R_{11}^2 p_- q_- + 2R_{22}^2 p_+ q_+ - 2R_{12}^2 p_- q_+ - 2R_{21}^2 p_+ q_- - 4R_{11}R_{12} p_- q_0 - 4R_{11}R_{21} p_0 q_- \\ & + 4R_{11}R_{22} p_0 q_0 + 4R_{12}R_{21} p_0 q_0 + 4R_{12}R_{22} p_0 q_+ + 4R_{21}R_{22} p_+ q_0 \\ & + 2(\boldsymbol{P}^2 + \boldsymbol{Q}^2)(I_2 + 1) - \boldsymbol{P}^4 - \boldsymbol{Q}^4 - 6\boldsymbol{P}^2\boldsymbol{Q}^2 + 4p_0\boldsymbol{Q}^2 + 4\boldsymbol{P}^2 q_0 - 4p_0^2 q_0. \end{aligned} \tag{4.5}$$

We denote by $p(p+1)$ and $q(q+1)$ the eigenvalues of operators $\boldsymbol{P}^2$ and $\boldsymbol{Q}^2$, respectively. As in Hughes (1983), we denote by P and Q the operators whose eigenvalues are p and q respectively. The eigenvalues of operators p_0 and q_0 will be denoted by α and β respectively. The projector onto $H^{pq}_{\alpha\beta}$ has the standard form

$$\prod{}^{pq}_{\alpha\beta}=\frac{1}{(2p+1)(2q+1)}\int \mathscr{D}^{p}_{\alpha\alpha}(u)\mathscr{D}^{q}_{\beta\beta}(v)U(u,v)\,\mathrm{d}u\,\mathrm{d}v \tag{4.6}$$

(no sum over α and β).

Proposition 4.1. If $x\in H_\infty$, then $\Pi^{pq}_{\alpha\beta}\,x\in H_\infty$.

The proof is accomplished in a standard way proceeding from formulae (1.2) and (4.6), the compactness of group K and the continuity of functions $\mathscr{D}^{p}_{\alpha\beta}(u)$.

Corollary. $H^{pq}_{\alpha\beta}\cap H_\infty$ is dense in $H^{pq}_{\alpha\beta}$.

In the Poincaré-invariant case the unitary operator connecting the space having a zero momentum with a space having any other fixed value of momentum exists if the Casimir operator of second order (the mass operator squared) has the positive lower bound (see, e.g., Lev 1984). At the same time, the representations describing tachyons and the particles with zero mass do not possess the 'rest' states. We shall see soon (proposition 4.2) that if the operator $W=-2I_2$ is chosen as the de Sitter analogue of the mass operator squared, then, taking into account our choice for the de Sitter analogue of the momentum operator, one has an analogous situation.

Since $\mathscr{D}(W)=H_\infty$, then it follows from proposition 4.1 that $W\Pi^{pq}_{\alpha\beta}\supset\Pi^{pq}_{\alpha\beta}W$. Hence, $\bar{W}\Pi^{pq}_{\alpha\beta}\supset\Pi^{pq}_{\alpha\beta}\bar{W}$. Let $w^{pq}_{\alpha\beta}$ and $\omega^{pq}_{\alpha\beta}$ be the reductions of W and $\bar{W}$, respectively, on $H^{pq}_{\alpha\beta}$. Then, one would easily see that $\omega^{pq}_{\alpha\beta}=\overline{w^{pq}_{\alpha\beta}}$. According to theorem 1.2, $\bar{W}$ is self-adjoint and thus $\omega^{pq}_{\alpha\beta}$ is the self-adjoint operator in $H^{pq}_{\alpha\beta}$. Analogously, one can consider the reductions of I_4, $\boldsymbol{P}^2$, $\boldsymbol{Q}^2$ on $H^{pq}_{\alpha\beta}$.

We define the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ shift operators by the same formulae as in Hughes' (1983) paper

$$\mathcal{O}^{++}=R_{11}(P+p_0+1)(Q+q_0+1)+R_{22}p_+q_++R_{12}(P+p_0+1)q_++R_{21}p_+(Q+q_0+1) \tag{4.7}$$

$$\mathcal{O}^{+-}=-R_{12}(P+p_0+1)(Q+q_0)+R_{21}p_+q_-+R_{11}(P+p_0+1)q_--R_{22}p_+(Q+q_0) \tag{4.8}$$

$$\mathcal{O}^{-+}=-R_{21}(P+p_0)(Q+q_0+1)+R_{12}\,p_-q_+-R_{22}(P+p_0)q_++R_{11}p_-(Q+q_0+1) \tag{4.9}$$

$$\mathcal{O}^{--}=-R_{22}(P+p_0)(Q+q_0)-R_{11}p_-q_-+R_{21}(P+p_0)q_-+R_{12}p_-(Q+q_0). \tag{4.10}$$

These operators are defined on H_∞; contrary to the work by Hughes (1983), we shall not use $A^{1/2,1/2}$, etc, operators since they can contain uncertainties and an improper handling of them would cause errors.

As in the paper by Hughes (1983), it follows from (1.3) and (4.7)-(4.10) that if $x\in H^{pq}_{\alpha\beta}\cap H_\infty$, then $\mathcal{O}^{++}x\in H^{p+1/2,q+1/2}_{\alpha+1/2,\beta+1/2}\cap H_\infty$, $\mathcal{O}^{+-}x\in H^{p+1/2,q-1/2}_{\alpha+1/2,\beta-1/2}\cap H_\infty$, $\mathcal{O}^{-+}x\in H^{p-1/2,q+1/2}_{\alpha-1/2,\beta+1/2}\cap H_\infty$, $\mathcal{O}^{--}x\in H^{p-1/2,q-1/2}_{\alpha-1/2,\beta-1/2}\cap H_\infty$ and on the space $H^{pq}_{\alpha\beta}\cap H_\infty$

$$\mathcal{O}^{++}\mathcal{O}^{--}=-\tfrac{1}{2}(p+\alpha)(q+\beta)\{I_4+(p+q)(p+q+1)[W+(p+q+2)(p+q-1)]\} \tag{4.11}$$

$$\mathcal{O}^{+-}\mathcal{O}^{-+}=\tfrac{1}{2}(p+\alpha)(q+\beta+1)\{I_4+(p-q)(p-q-1)[W+(p-q+1)(p-q-2)]\} \tag{4.12}$$

$$\mathcal{O}^{-+}\mathcal{O}^{+-}=\tfrac{1}{2}(p+\alpha+1)(q+\beta)\{I_4+(q-p)(q-p-1)[W+(q-p+1)(q-p-2)]\} \tag{4.13}$$

$$\mathcal{O}^{--}\mathcal{O}^{++}=-\tfrac{1}{2}(p+\alpha+1)(q+\beta+1)\{I_4+(p+q+1)(p+q+2)[W+(p+q)(p+q+3)]\}. \tag{4.14}$$

Proceeding from theorem 1.1 and formulae (1.3) and (4.3), one can show in analogy with Hughes (1983) that, if $x \in H^{pq}_{\alpha\beta} \cap H_\infty$, then

$$(x, \mathcal{O}^{++}\mathcal{O}^{--}x) = -\frac{pq}{(p+\frac{1}{2})(q+\frac{1}{2})}\| \mathcal{O}^{--}x \|^2 \tag{4.15}$$

$$(x, \mathcal{O}^{+-}\mathcal{O}^{-+}x) = -\frac{p(q+1)}{(p+\frac{1}{2})(q+\frac{1}{2})}\| \mathcal{O}^{-+}x \|^2 \tag{4.16}$$

$$(x, \mathcal{O}^{-+}\mathcal{O}^{+-}x) = -\frac{(p+1)q}{(p+\frac{1}{2})(q+\frac{1}{2})}\| \mathcal{O}^{+-}x \|^2 \tag{4.17}$$

$$(x, \mathcal{O}^{--}\mathcal{O}^{++}x) = -\frac{(p+1)(q+1)}{(p+\frac{1}{2})(q+\frac{1}{2})}\| \mathcal{O}^{++}x \|^2. \tag{4.18}$$

Let $H(p=0)$ be a direct sum of the spaces H^{0q} with all possible q, and $H(q=0)$ be defined analogously. Suppose that the lower bound of the operator $\bar{W}$ is positive, and this condition will be written as $\bar{W} > 0$.

Proposition 4.2 Let $\bar{W} > 0$. Then we have either $H(p=0) \neq 0$ or $H(q=0) \neq 0$.

Proof. Suppose that the opposite situation takes place, i.e. $H(p=0) = H(q=0) = 0$. Let $H^{\underline{p}q'}_{\alpha\beta}$ be the non-zero space with the minimum $p = \underline{p}$ and some $q = q'$. According to our assumption, $\underline{p}$, $q' \neq 0$. From the properties of operators $\mathcal{O}$ and from (4.11) and (4.12) it follows that if $\alpha \neq -\underline{p}$, $\beta \neq -q'$ then we have on $H^{\underline{p}q'}_{\alpha\beta} \cap H_\infty$

$$\begin{aligned} &I_4 + (\underline{p}+q')(\underline{p}+q'+1)[W + (\underline{p}+q'+2)(\underline{p}+q'-1)] = 0 \\ &I_4 + (\underline{p}-q')(\underline{p}-q'-1)[W + (\underline{p}-q'+1)(\underline{p}-q'-2)] = 0. \end{aligned} \tag{4.19}$$

Since I_4 and W commute with p_- and q_-, these relations take place at all $\alpha = -\underline{p}, \ldots, \underline{p}$ and $\beta = -q', \ldots, q'$. As in Hughes (1983), elimination of I_4 gives

$$w^{pq}_{\alpha\beta} = -2[\underline{p}^2 + q'(q'+1) - 1] \tag{4.20}$$

Hence, if $\underline{p}$, $q' \neq 0$, then $\omega^{\underline{p}q'}_{\alpha\beta} \leq 0$, which contradicts our assumption.

The condition $\bar{W} > 0$ is a sufficient one for the existence of the 'rest' states. Indeed, as is obvious from §§ 2 and 3, the 'rest' state always exists for any system of particles corresponding to the principal series of UIR. In the case of UIR, calculation of W proceeding from (2.2) or (2.4) gives the well known result

$$W = \mu^2 - s^2 + \tfrac{9}{4}. \tag{4.21}$$

Therefore, $\bar{W} < 0$ is possible for the principal series representations. However, if spin is zero, the condition $\bar{W} > 0$ is also a necessary one (see below).

In the following part of this section we confine ourselves to consideration of systems with spin zero, since the case of a non-zero spin is essentially more complicated from the technical point of view.

Let again $H(p=0) \neq 0$. By definition of the spin operators, the spin is zero if $\boldsymbol{M}x = 0$ $\forall x \in H(p=0) \cap H_\infty$, since the action of the spin operator on the 'rest' states is defined as the action of operator $\boldsymbol{M}$ on these states. At the same time, by definition of the space $H(p=0)$, $\boldsymbol{P}x = 0$ for such x and, therefore, $\boldsymbol{Q}x = 0$ (see (4.1)). The case $H(q=0) \neq 0$ can be considered in a similar way. Combining this result with proposition 4.2, we have that if the following condition 4.3 is satisfied then the following proposition 4.4 occurs.

Condition 4.3. The representation $g \to U(g)$ has the property $\bar{W} > 0$ and is the representation with spin zero.

Proposition 4.4. $H^0 = H(p=0) = H(q=0) \neq 0$ and $\forall x \in H^0 \cap H_\infty$, $M_{ij}x = 0$ for all i, j.

From the continuity of operators $U(g)$ one can now easily prove the following proposition 4.5.

Proposition 4.5. Let ϕ be a set dense in H^0. Then at the condition 4.3 the linear span of the elements $U(g)x$, $g \in \overline{\mathrm{SO}_0(1,4)}$, $x \in \phi$ is dense in H.

It is evident from proposition 4.4 that $U(k)x = x$ if $k \in \mathrm{K}$, $x \in H^0$. Therefore, as easily follows from proposition 4.5, the linear span of the elements $U(\exp(t_j L^{0j}))x$, $x \in \phi$ is dense in H.

Let $\mathcal{M}^2$ be the reduction of $\bar{W}$ on H^0. This notation shows that the operator $\mathcal{M} \equiv (\mathcal{M}^2)^{1/2}$ is chosen as the de Sitter analogue of a mass operator. Let $e(\lambda)$ be the spectral function of operator $\mathcal{M}$ and the role of ϕ be taken by $x \in H^0$ satisfying the condition $e(\Delta)x = x$ for some finite interval Δ. Then, using theorem 3 from § 7 of Nelson (1959), one can prove that every $x \in \phi$ belongs also to H_∞ and the series

$$U(\exp(t_j L^{0j}))x = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} t_{j_1} \ldots t_{j_n} M^{0j_1} \ldots M^{0j_n} x \qquad x \in \phi \tag{4.22}$$

is convergent for any 4-vector $\{t_j\}$. Let $R_{j_1 \ldots j_n}$ be obtained from $M^{0j_1} \ldots M^{0j_n}$ by the symmetrisation over all indices (since SO(4) is the orthogonal group, the upper and lower indices are equivalent). Then, taking into account the formula (4.22) and proposition 4.5, we have the following proposition 4.6.

Proposition 4.6. Under condition 4.3, the linear span of elements x, $t_{j_1 \ldots j_n} R_{j_1 \ldots j_n} x$ $(x \in H^0 \cap H_\infty,\ n = 1, 2, \ldots)$ is dense in H.

The symmetric tensor operator $R_{j_1 \ldots j_n}$ is in one-to-one correspondence with the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ spinorial operator $R_{\rho_1 \ldots \rho_n, \sigma_1 \ldots \sigma_n}$ which is symmetric relative to any permutation of pairs (ρ_i, σ_i). This operator can be decomposed into irreducible components

$$R_{\rho_1 \ldots \rho_n, \sigma_1 \ldots \sigma_n} = \sum_{pq} R^{pq}_{\rho_1 \ldots \rho_n, \sigma_1 \ldots \sigma_n} \tag{4.23}$$

where p and q range independently from $0, \frac{1}{2}, \ldots, \frac{1}{2}n$. From the above-mentioned symmetry property one can see that in fact only the terms with $p = q = J$ are present in the sum (4.23). Therefore, from propositions 4.1 and 4.6 and theorem 1.2, there follows proposition 4.7.

Proposition 4.7. Under condition 4.3 the decomposition

$$H = \sum_{pq\alpha\beta} \oplus H^{pq}_{\alpha\beta} \tag{4.24}$$

contains only the spaces with $p = q$.

Proposition 4.8. Let the decomposition (4.24) contain only the spaces with $p = q$. Then $\bar{I}_4 = 0$.

Proof. It is sufficient to consider the action of I_4 on every subspace $H^{pp}_{\alpha\beta} \cap H_\infty$. At $p=0$, $V^a=0$ as follows from proposition 4.4. Let now $p \neq 0$. Since the operators (4.12) and (4.13) should be equal to zero, then $\bar{I}_4 = 0$ if $\alpha, \beta \neq -p$. Since I_4 commutes with p_- and q_-, then $\bar{I}_4 = 0$ everywhere.

It follows from (4.15)-(4.18) that the operators (4.11)-(4.14) are negative semidefinite. Under the conditions of proposition 4.8 we have from (4.14) and (4.18) that

$$\omega^{pp}_{\alpha\beta} + 2p(2p+3) \geq 0 \tag{4.25}$$

and, in addition, if $p \neq 0$, then it follows from (4.11) and (4.15) that

$$\omega^{pp}_{\alpha\beta} + 2(p+1)(2p-1) \geq 0 \tag{4.26}$$

and the other conditions are satisfied automatically.

We have defined the notion of spin only for the systems having a 'rest' state. We have shown in this case that proposition 4.7 is valid for systems with spin zero. Alternatively, we might define a system with spin zero as one for which proposition 4.7 holds. Such a definition is less obvious but is applicable also for the systems which have no 'rest' state. In any case, as follows from proposition 4.8, formulae (4.25) and (4.26) are valid.

It follows from (4.25) that, if $\bar{W} < 0$, then $p > 0$, i.e. the systems with $\bar{W} < 0$ cannot have a 'rest' state as in the Poincaré-invariant theory. As follows from (4.21), the case $\bar{W} = 0$ corresponds to the UIR with $i\mu = \frac{3}{2}$. It can be shown that this representation (belonging to the discrete series) also has no 'rest' state, but we shall not dwell on this. We shall consider in more detail the case of UIR. It is known that for $s=0$ the values $\bar{W} < 0$ are obtained in the case of the discrete series with $i\mu = n + \frac{3}{2}$ $(n = 1, 2, \ldots)$. The condition (4.26) is in this case stronger than (4.25), and we have from (4.21)

$$p - \tfrac{1}{2}(n+1) \geq 0. \tag{4.27}$$

This corresponds to the fact that representations of the discrete series are interpreted as tachyons.

In the remaining part of this section we suppose without reservations that the condition 4.3 is satisfied. In this case, the decomposition of H on the subspaces with different 'momenta' (see § 3) is the standard $SU(2) \times SU(2)$ decomposition of H and $H^J_{\alpha\beta} = H^{pp}_{\alpha\beta}$.

Let us consider the term $R^{n/2, n/2}_{\rho_1 \ldots \rho_n, \sigma_1 \ldots \sigma_n}$ in the decomposition (4.23). This is the irreducible $SU(2) \times SU(2)$ spinor which is completely symmetric in indices $\rho_1, \ldots, \rho_n$ as well as in indices $\sigma_1, \ldots, \sigma_n$. It can be represented as $R^J_{\alpha\beta}$ $(J = n/2)$, where α is defined simply by the number of unities in the set $\{\rho_1, \ldots, \rho_n\}$ and β is defined in a similar way. The operator $R_J = R^J_{\rho_1 \ldots \rho_n, \sigma_1 \ldots \sigma_n} R^{J, \rho_1 \ldots \rho_n, \sigma_1 \ldots \sigma_n}$ commutes with all M_{ij}, and thus, as follows from theorems 1.1, 1.2 and formula (4.3), $\bar{R}_J$ is self-adjoint and $\bar{R}_J \geq 0$. Let r_J be the reduction of R_J on H^0. Proceeding from theorem 1.1, proposition 4.8 and formulae (4.3), (4.7), (4.14) and (4.18), one can prove the following proposition 4.9.

Proposition 4.9. $r_J > 0$.

Therefore, one can define $r_J^{-1/2}$, and this operator is bounded. Let us consider the set $\phi_J = \{r_J^{1/2} x, x \in H^0 \cap H_\infty\}$. Since $H^0 \cap H_\infty$ is the kernel of operator r_J and $\mathscr{D}(r_J)$ is the kernel of operator $r_J^{1/2}$, it follows readily from proposition 4.9 that ϕ_J is dense in H^0. One can check that the operators

$$\mathcal{U}^J_{\alpha\beta} = (2J+1)(C_{2J}^{J+\alpha} C_{2J}^{J+\beta})^{1/2} R^J_{\alpha\beta} r_J^{-1/2} \tag{4.28}$$

(by C_n^m we denote binomial coefficients) map isometrically ϕ_J into $H^J_{\alpha\beta}$. Therefore, $\mathcal{U}^J_{\alpha\beta}$ can be extended to the whole space H^0, and by means of proposition 4.6 one can prove the following proposition 4.10.

Proposition 4.10. The operators $\mathcal{U}^J_{\alpha\beta}$ are unitary operators from H^0 to $H^J_{\alpha\beta}$.

The operator $\mathcal{U}^{-1} = \Sigma_{J\alpha\beta} \oplus (\mathcal{U}^J_{\alpha\beta})^{-1}$ is the unitary operator from H to $L_2(J, \alpha, \beta) \otimes H^0$. The elements x of the latter space are defined by their components $x^J_{\alpha\beta} \in H^0$ which satisfy the condition

$$\sum_{J\alpha\beta} \| x^J_{\alpha\beta} \|^2 < \infty. \tag{4.29}$$

It can be readily shown that the action of representation generators of the group K in the space $L_2(J, \alpha, \beta) \otimes H^0$ is defined by the standard formulae

$$\begin{aligned} (p_+x)^J_{\alpha\beta} &= [(J+\alpha)(J-\alpha+1)]^{1/2} x^J_{\alpha-1,\beta} \qquad & (p_-x)^J_{\alpha\beta} &= [(J+\alpha+1)(J-\alpha)]^{1/2} x^J_{\alpha+1,\beta} \\ (q_+x)^J_{\alpha\beta} &= [(J+\beta)(J-\beta+1)]^{1/2} x^J_{\alpha,\beta-1} \qquad & (q_-x)^J_{\alpha\beta} &= [(J+\beta+1)(J-\beta)]^{1/2} x^J_{\alpha,\beta+1} \end{aligned} \tag{4.30}$$

$$(p_0x)^J_{\alpha\beta} = \alpha x^J_{\alpha\beta} \qquad (q_0x)^J_{\alpha\beta} = \beta x^J_{\alpha\beta}$$

and the action of operators $R_{\rho\sigma}$ can be represented in the form

$$\begin{aligned} (R_{\rho\sigma}x)^J_{\alpha\beta} &= a(J, \alpha, \beta, \rho, \sigma) r_J^{-1/2} r_{J+1/2}^{1/2} x^{J+1/2}_{\alpha-\rho,\beta-\sigma} \\ &\quad + b(J, \alpha, \beta, \rho, \sigma) r_J^{1/2} r_{J-1/2}^{-1/2} x^{J-1/2}_{\alpha-\rho,\beta-\sigma}. \end{aligned} \tag{4.31}$$

The coefficients $a(J, \alpha, \beta, \rho, \sigma)$ and $b(J, \alpha, \beta, \rho, \sigma)$ can be calculated in principle, but we have no need to know their explicit form. In addition, it can be shown that r_J is the polynomial of $2J$th order of $\mathcal{M}^2$. Hence the results of our consideration can be formulated in the form of the following decomposition theorem.

Theorem 4.11. Let $g \to U(g)$ be the continuous unitary representation of the group $\overline{\mathrm{SO}_0(1,4)}$ in the Hilbert space H and $W = -\frac{1}{2} M_{ab} M^{ab}$. If $\bar{W} > 0$ and the spin is equal to zero, then the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ decomposition $H = \Sigma_{J\alpha\beta} \oplus H^J_{\alpha\beta}$ contains only subspaces with $p = q = J$. The operators $\mathcal{U}^J_{\alpha\beta}$ from H^0 to $H^J_{\alpha\beta}$ are unitary and given by (4.28) on the sets dense in H^0. Representation generators of the subgroup $\mathrm{SU}(2) \times \mathrm{SU}(2)$ in the space $L_2(J, \alpha, \beta) \otimes H^0 = (\Sigma_{J\alpha\beta} \oplus (\mathcal{U}^J_{\alpha\beta})^{-1})H$ have the standard form (4.30) while the action of operators M^{0i} in this space is fully defined by the operator $\mathcal{M}^2$ which is the reduction of $\bar{W}$ on H^0.

The method of packing operators developed by Sokolov (1977, 1978) (see also the papers by Coester and Polyzou (1982), Mutze (1984) and Lev (1984, 1985)) shows that in the Poincaré-invariant theory one can divide variables into 'external' and 'internal' ones, and all interactions are defined by the mass operator in the 'internal' space. According to theorem 4.11, this is also the case in $\overline{\mathrm{SO}_0(1,4)}$-invariant theory, at least for the systems with spin zero. Hence, we can guarantee that the cluster separability (see § 1) holds if the introduction of interaction into the system is accomplished as usual through the introduction of some interaction operators into the free-mass operator.

5. The mass operator of a system of two particles

Taking into account the remarks made at the end of the preceding section, it is natural to consider first the properties of free-mass operators. In the case of free particles 1 and 2 the corresponding representation of the $\overline{SO_0(1,4)}$ is equal to the tensor product of the corresponding single-particle representations. Proceeding from the formula (2.4) and results of § 3, one can calculate explicitly the operator $\mathcal{M}^2$ which, as well as in the spinless case, can be defined as the reduction of W on the subspace $H(p=0)$. The result of a simple but time-consuming calculation is as follows:

$$\mathcal{M}^2 = [(1/\delta_4)\boldsymbol{l}(\boldsymbol{\delta}) + \delta_4 \boldsymbol{S}_0 + (\mu_1-\mu_2)\boldsymbol{\delta} - \boldsymbol{\delta}\times(\boldsymbol{s}_1-\boldsymbol{s}_2)]^2 + \{(\mu_1+\mu_2)\delta_4 + \mathrm{i}[\delta_4(\boldsymbol{\delta}(\partial/\partial\boldsymbol{\delta})+\tfrac{3}{2}) - \delta^2/\delta_4] - (\boldsymbol{\delta}, \boldsymbol{s}_1-\boldsymbol{s}_2)\}^2 + \tfrac{1}{4}(9+\boldsymbol{\delta}^2) - 2\boldsymbol{S}^2 \tag{5.1}$$

where $\boldsymbol{S}_0 = \boldsymbol{s}_1 + \boldsymbol{s}_2$, $\boldsymbol{S} = \boldsymbol{S}_0 + \boldsymbol{l}(\boldsymbol{\delta})$, $\boldsymbol{\delta} = \tilde{\boldsymbol{n}}_1$ is chosen as the 'internal' variable, and $\delta_4 = (1-\boldsymbol{\delta}^2)^{1/2}$. The role of the 'internal' volume element (3.5) in the considered case is taken on by

$$8\delta_4\, \mathrm{d}^3\boldsymbol{\delta} = 8\delta_4\delta^2\, \mathrm{d}\delta\, \mathrm{d}o \tag{5.2}$$

where $\delta = |\boldsymbol{\delta}|$, and $\mathrm{d}o$ is an element of the spatial angle.

Let us proceed to the space of functions quadratically integrable over the measure $\mathrm{d}\delta\, \mathrm{d}o/\delta_4\delta$, introduce the variables

$$r = -\ln\{[1-(1-\delta^2)^{1/2}]/\delta\} \qquad \boldsymbol{\xi} = \boldsymbol{\delta}/\delta \tag{5.3}$$

and perform the unitary transformation by multiplication of all functions by

$$\exp\left(\mathrm{i}(\mu_1+\mu_2)\int_0^r \tanh t\, \mathrm{d}t\right).$$

As a result, we have that $\mathcal{M}^2$ can be realised as

$$\mathcal{M}^2 = \left(\frac{1}{\tanh r}\boldsymbol{l}(\boldsymbol{\xi}) + \tanh r(\boldsymbol{s}_1+\boldsymbol{s}_2) + \frac{(\mu_1-\mu_2)\boldsymbol{\xi}}{\cosh r} - \frac{\boldsymbol{\xi}\times(\boldsymbol{s}_1-\boldsymbol{s}_2)}{\cosh r}\right)^2 + \left(-\mathrm{i}\frac{\partial}{\partial r} - \frac{(\boldsymbol{\xi}, \boldsymbol{s}_1-\boldsymbol{s}_2)}{\cosh r}\right)^2 + \frac{1}{4\cosh^2 r} + \frac{9}{4} - 2\boldsymbol{S}^2 \tag{5.4}$$

in the space of spinorial functions $\varphi(r, \boldsymbol{\xi})$ such that

$$\int \|\varphi(r, \boldsymbol{\xi})\|^2\, \mathrm{d}r\, \mathrm{d}o < \infty. \tag{5.5}$$

Performing another unitary transformation

$$\varphi(r, \boldsymbol{\xi}) \to \exp\left(\mathrm{i}(\boldsymbol{\xi}, \boldsymbol{s}_1-\boldsymbol{s}_2)\int_0^r \frac{\mathrm{d}t}{\cosh t}\right)\varphi(r, \boldsymbol{\xi}) \tag{5.6}$$

and taking into account

$$\exp[-\mathrm{i}(\boldsymbol{\xi s})\omega]\boldsymbol{s}\exp[\mathrm{i}(\boldsymbol{\xi s})\omega] = \boldsymbol{s}\cos\omega + \boldsymbol{\xi}(\boldsymbol{\xi s})(1-\cos\omega) + (\boldsymbol{s}\times\boldsymbol{\xi})\sin\omega \tag{5.7}$$

one obtains after simple calculations

$$\widetilde{\mathcal{M}^2} = -\frac{\partial^2}{\partial r^2} + \frac{\cosh r}{\sinh^2 r}\boldsymbol{l}(\boldsymbol{\xi})^2 - \frac{\cosh r - 1}{\sinh^2 r}(\boldsymbol{S}^2 + \boldsymbol{S}_0^2) + \frac{(\cosh r - 1)^2(1+2\cosh r)}{\sinh^2 r\cosh^2 r}(\boldsymbol{\xi S}_0)^2 + \frac{2\sinh r}{\cosh^2 r}(\mu_1-\mu_2)(\boldsymbol{\xi S}_0) + \frac{(\mu_1-\mu_2)^2+\frac{1}{4}}{\cosh^2 r}$$

$$\mathcal{M}^2 = \widetilde{\mathcal{M}^2} - \boldsymbol{S}^2 + \tfrac{9}{4}. \tag{5.8}$$

Having an explicit form for the mass operator, one can determine, in particular, which UIR are contained in the decomposition of the tensor product of representations corresponding to the principal series. The spectrum of operator $\widetilde{\mathcal{M}^2}$ can be studied in the same manner as that for the non-relativistic Schrödinger operator. At $r \to 0$ we have that

$$\widetilde{\mathcal{M}^2} \approx -\partial^2/\partial r^2 + l(\xi)^2/r^2 \tag{5.9}$$

as well as for a free Hamiltonian. Therefore, the situation known as the 'fall onto the centre' does not occur, and the operator $\widetilde{\mathcal{M}^2}$ necessarily contains the continuous spectrum ranging from $(0, \infty)$. Comparing (4.21) with (5.8), we conclude that the following theorem takes place.

Theorem 5.1. Let $g \to U_1(g)$ and $g \to U_2(g)$ be UIR of the group $\overline{\mathrm{SO}_0(1,4)}$ belonging to the principal series. Then the decomposition of their tensor product into UIR necessarily contains UIR of the principal series with all possible spins s (either integer or half-integer depending on whether $s_1 + s_2$ is an integer or a half-integer) and all $\mu \in (0, \infty)$.

Theorem 5.1 can be proved in principle by a more straightforward way using Mackey's theorem on a tensor product of induced representations (see, e.g., ch 18 of the book by Barut and Raczka (1977)). In the given case one should take the representation of the subgroup ASU(2):

$$\Delta(\tau_A r) = \exp[\mathrm{i}(\mu_1 - \mu_2)\tau]\Delta_{s_1}(r) \otimes \Delta_{s_2}(r) \tag{5.10}$$

induce it in $\overline{\mathrm{SO}_0(1,4)}$ and then decompose into UIR. The latter task is not an easy one from the technical point of view. In any case, the knowledge of the explicit form of the mass operator gives information not only on decomposition of the tensor product into UIR but also on other properties of a two-particle system (for example, relative motion of particles).

As follows from theorem 5.1, for any system of free particles corresponding to the principal series of UIR the spectrum of the mass operator contains all $\mu \in (0, \infty)$. This result has no analogy either in the Poincaré-invariant or SO(2, 3)-invariant theories where the mass can be defined as the lowest value of energy. Therefore, the spectrum of the mass operator of N particles with masses $m_1, \ldots, m_N$ is bounded here from below by the value $(m_1 + \ldots + m_N)$.

Let us give a simple example illustrating theorem 5.1. Let the spinless particles 1 and 2 be in a rest state. Let M_1^{ab} and M_2^{ab} be the corresponding representation generators, and φ_1 and φ_2 be the corresponding normalised wavefunctions. Then (see proposition 4.4) $M_1^{ij}\varphi_1 = 0$ at all i, j, and since φ_1 is the SO(4) scalar we have that $(\varphi_1, M_1^{0i}\varphi_1) = 0$ at all i. Analogous relations hold for the second particle. Therefore, an average value of the operator W

$$\langle W \rangle = \langle \varphi_1\varphi_2 | -\tfrac{1}{2}(M_1^{ab} + M_2^{ab})(M_{1ab} + M_{2ab}) | \varphi_1\varphi_2 \rangle \tag{5.11}$$

satisfies the equality $\langle W \rangle = \langle \varphi_1 | W_1 | \varphi_1 \rangle + \langle \varphi_2 | W_2 | \varphi_2 \rangle$. Hence, for an average value of μ^2 (see (4.21)) we have

$$\langle \mu^2 \rangle = \mu_1^2 + \mu_2^2 + \tfrac{9}{4} \tag{5.12}$$

and if at the contraction to the Poincaré group the particles remain massive (i.e. the values of μ_1 and μ_2 are sufficiently large) then the above value is evidently less than $(\mu_1 + \mu_2)^2$.

6. The de Sitter correction in a non-relativistic approximation

In order to understand theorem 5.1 on a 'more physical level', let us consider again the generators (2.2). Let velocities of the particles be low ($v \ll 1$). Then we can easily calculate the correction of the order $1/R$ to the conventional operators $\boldsymbol{P}$ and E. We have

$$\boldsymbol{P}_j = \boldsymbol{p}_j + \frac{\mathrm{i}m_j}{R}\frac{\partial}{\partial \boldsymbol{p}_j} \qquad E_j = m_j + \frac{\boldsymbol{p}_j^2}{2m_j} + \frac{\mathrm{i}}{R}\left(\boldsymbol{p}_j\frac{\partial}{\partial \boldsymbol{p}_j} + \frac{3}{2}\right) \tag{6.1}$$

where $\boldsymbol{p}_j = m_j\boldsymbol{v}_j$, $j = 1, 2$, and there is no sum over j. We introduce for the two-particle system the well known quantities

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 \qquad \boldsymbol{q} = \frac{m_2\boldsymbol{p}_1 - m_1\boldsymbol{p}_2}{m_1 + m_2}. \tag{6.2}$$

Then we can express the total momentum and energy in the form

$$\boldsymbol{P}_1 + \boldsymbol{P}_2 = \boldsymbol{P} + \frac{\mathrm{i}M}{R}\frac{\partial}{\partial \boldsymbol{P}} \qquad E_1 + E_2 = M + \frac{\boldsymbol{P}^2}{2M} + \frac{\mathrm{i}}{R}\left(\boldsymbol{P}\frac{\partial}{\partial \boldsymbol{P}} + \frac{3}{2}\right) \tag{6.3}$$

where

$$M = m_1 + m_2 + \frac{m_{12}\boldsymbol{q}^2}{2} + \frac{\mathrm{i}}{R}\left(\boldsymbol{q}\frac{\partial}{\partial \boldsymbol{q}} + \frac{3}{2}\right) \tag{6.4}$$

and m_{12} is a reduced mass of particles 1 and 2. Comparing (6.1) with (6.3) and (6.4), we see that a due account of the de Sitter correction results in an effective interaction in the mass operator. Since $\boldsymbol{q}$ is the non-relativistic relative momentum, a relative radius vector can be introduced through a conventional procedure assuming $\boldsymbol{r} = \mathrm{i}\partial/\partial\boldsymbol{q}$. In such a coordinate representation the non-relativistic internal energy operator in spherical coordinates is

$$H_{\mathrm{NR}} = -\frac{1}{2m_{12}}\Delta - \frac{\mathrm{i}}{R}\left(r\frac{\partial}{\partial r} + \tfrac{3}{2}\right) \tag{6.5}$$

where $r = |\boldsymbol{r}|$. We see that an effective interaction between the particles is described by the interaction operator

$$V = -\frac{\mathrm{i}}{R}\left(r\frac{\partial}{\partial r} + \frac{3}{2}\right). \tag{6.6}$$

Let us determine the spectrum of operator (6.5). When doing so, it should be noted that at sufficiently long distances $\Delta/2m_{12}$ is negligible compared with V. Hence, solving the equation $H_{\mathrm{NR}}\psi = \lambda\psi$, we see that at long distances the function ψ depends on r in the following way:

$$\psi(r) \sim r^{-3/2 + \mathrm{i}\lambda R}. \tag{6.7}$$

Therefore, there exist solutions at any λ from the interval $(-\infty, \infty)$, and thus the spectrum of the operator H_{NR} occupies this interval. Moreover, since the functions (6.7) are not quadratically integrable, the whole spectrum is continuous. Analogously, one can easily show that for the system with an arbitrary number of particles the spectrum of operator H_{NR} also contains the whole interval $(-\infty, \infty)$.

The unboundedness of the 'energy spectrum' (the spectrum of operator M_{04}) from below is well known, being, probably, the basic feature distinguishing the $\overline{\mathrm{SO}_0(1,4)}$-invariant theory from the $\overline{\mathrm{SO}_0(2,3)}$-invariant one. As we have shown above, the same holds also for the Hamiltonian of the 'internal' motion.

In the case considered, introduction of interaction into a system can be accomplished in a conventional way, i.e. by adding together the interaction operator and H_{NR}. If the interactions introduced are the conventional Coulomb or nuclear ones, then at long distances they are negligible when compared with (6.6). Therefore, the mass operator still has a purely continuous spectrum ranging from $(-\infty, \infty)$, and there are no bound states in the theory. This condition by no means contradicts the experience, since the actual lifetime of a quasistationary state can be very long.

The qualitative explanation of the above result is as follows. It is known that the de Sitter world possesses antigravity: the force of repulsion between particles is proportional to the distance between them. In the quantum case, this interaction is so strong that it results in a complete rearrangement of the spectrum of the non-perturbed mass operator. One may naturally suppose that the operator (6.6) is just a quantum operator corresponding to the universal classical repulsion. Indeed, the classical Hamilton function for radial motion, corresponding to the operator (6.5), is

$$H(p, r) = p^2/2m + pr/R. \tag{6.8}$$

Solving the canonical equations of motion, we have

$$r(t) = c_1 \exp(t/R) + c_2 \exp(-t/R) \tag{6.9}$$

where c_1 and c_2 are arbitrary constants. It follows from (6.9) that $\ddot{r} = r/R^2$, as it should be.

7. Conclusion

The main results of this work are theorems 4.11 and 5.1. The former allows for interpretation of the momentum and boost operators in analogy with the Poincaré-invariant case. Until now, the different de Sitter analogue of the momentum operator has been dealt with, and it has been noted (see, e.g., Moylan 1983) that the physical meaning of the de Sitter boosts is still unclear. Note, however, that one cannot achieve a complete correspondence with the Poincaré-invariant case at any choice of the de Sitter analogues. For example, one cannot define the mass in the SO(1, 4) case as the lowest value of energy.

Theorem 4.11 has been proved only for the case of spin zero. Apart from the general properties of unitary representations of groups, only the commutation relations (1.3) (and Hughes' method of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ shift operators which follows from them) have been used. It can be assumed that an analogous decomposition exists also at $S \neq 0$ but the proof in this case is more complicated and requires utilisation of finer properties of representations of the group $\overline{\mathrm{SO}_0(1,4)}$.

The decomposition described in theorem 4.11 is in some aspects similar to the conventional decomposition of unitary representations into UIR. However, theorem 4.11 describes the decomposition in more detail; in particular, it defines explicitly the unitary operator realising H as $L_2(J, \alpha, \beta) \otimes H^0$.

The result of theorem 5.1 implies that even for the non-interacting particles with the 'masses' μ_1 and μ_2 the system mass operator contains all μ from 0 to ∞, and this result would seem unexpected. This result has been explained in § 6 where the quantum operator corresponding to the de Sitter antigravity has been obtained from the $\overline{SO_0(1,4)}$ invariance only (without any assumptions concerning the locality) while the constant of interaction $1/R$ arises only when an interaction is interpreted in the Poincaré-invariant terms. In our opinion, this example poses the following question: can all the existing interactions be interpreted as a result of transition from the high symmetry to the lower one?

References

Barut A O and Raczka R 1977 *Theory of Group Representations and Applications* (Warsaw: PWN)
Coester F and Polyzou W N 1982 *Phys. Rev.* D **26** 1348-67
Dixmier J 1961 *Bull. Soc. Math. Fr.* **89** 9-41
Evans N T 1967 *J. Math. Phys.* **8** 170-84
Fabec R C 1977 *J. Math. Phys.* **18** 2087-91
Fronsdal C 1965 *Rev. Mod. Phys.* **37** 221-4
Haag R and Kastler D 1964 *J. Math. Phys.* **5** 848-61
Hughes J W B 1983 *J. Math. Phys.* **24** 1015-20
Klimyk A U and Kachurik I I 1986 *Computational Methods in Group Representation Theory* (Kiev: Vyshcha shkola) (in Russian)
Lev F M 1984 *J. Phys. A: Math. Gen.* **17** 2047-58
—— 1985 *Nucl. Phys.* A **433** 605-18
Lev F M and Mirmovich E G 1984 unpublished
Mensky M B 1976 *Method of Induced Representations. Space-Time and the Concept of Particles* (Moscow: Nauka) (in Russian)
Moylan P 1983 *J. Math. Phys.* **24** 2706-21
—— 1985 *J. Math. Phys.* **26** 29-40
Mutze U 1984 *Phys. Rev.* D **29** 2255-69
Nelson E 1959 *Ann. Math.* **70** 572-615
Sokolov S N 1977 *Dokl. Akad. Nauk* **233** 575-8
—— 1978 *Teor. Mat. Fiz.* **36** 193-207
Strom S 1970 *Ann. Inst. H Poincaré* A **13** 77-98